MATH 521A: Abstract Algebra
Preparation for Exam 2

1. Find all solutions to the modular equation $50x \equiv 20 \pmod{830}$.
2. Determine, with proof, all zero divisors in $\mathbb{Z}_{33}$. How many are there?
3. Let p be prime. Determine, with proof, how many elements of $\mathbb{Z}_{p^2}$ are units.
4. Consider the function $f : \mathbb{Z}_{33} \to \mathbb{Z}_3 \times \mathbb{Z}_{11}$ given by $f : [x]_{33} \mapsto ([x]_3, [x]_{11})$. Prove or disprove that f is well-defined.
5. Consider the function $f : \mathbb{Z}_{33} \to \mathbb{Z}_3 \times \mathbb{Z}_{11}$ given by $f : [x]_{33} \mapsto ([x]_3, [x]_{11})$. Prove or disprove that f is a ring isomorphism.
6. Consider the function $f : \mathbb{Z}_{27} \to \mathbb{Z}_3 \times \mathbb{Z}_9$ given by $f : [x]_{27} \mapsto ([x]_3, [x]_9)$. Prove or disprove that f is well-defined, and a ring isomorphism.
7. Let p be prime, and let $[x], [y] \in \mathbb{Z}_p$. Prove that if $[x][y] = [0]$, then $[x] = [0]$ or $[y] = [0]$.
8. Let R have ground set $\mathbb{Z}$ and operations given by:
$$\forall x, y \in \mathbb{Z}, \quad x \oplus y = x + y - 2, \quad x \odot y = 2x + 2y - xy - 2.$$
Prove that R, with operations $\oplus, \odot$, is an integral domain.
9. Let R be a (not necessarily commutative) ring with identity and $x, y \in R$. Suppose that neither x nor y is a zero divisor, and that xy is a unit. Prove that x, y are units.
10. For ring R and element $x \in R$, we say that x is *silver* if $x+x+x+x+x = 0_R$. Define $T \subseteq R$ to be the set of silver elements of R. Prove that T is a subring of R.
11. Set $R = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$. Prove that R is a subring of $\mathbb{R}$, and find its units.
12. Set $R = \{a + b\sqrt{-1} : a, b \in \mathbb{Z}\}$. (We often call $\sqrt{-1}$ by the name i). Prove that R is a subring of $\mathbb{C}$, and find its units.
13. Let R be the set of 2×2 lower triangular matrices with entries from $\mathbb{Q}$, i.e. $R = \{\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) : a, b, c \in \mathbb{Q}\}$. Prove that R is a subring of the ring of 2×2 matrices with entries from $\mathbb{Q}$.
14. Let R be the ring of 2×2 lower triangular matrices with entries from $\mathbb{Q}$, i.e. $R = \{\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) : a, b, c \in \mathbb{Q}\}$. Determine, with proof, all units and zero divisors of R.
15. Let R be the ring of 2×2 matrices with entries from $\mathbb{Q}$. Define $f : R \to R$ via $f : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$, a.k.a. the matrix transpose. Prove or disprove that f is a ring isomorphism.